New Methods for Suboptimal Feedback Control of Parabolic Systems

A new class of algorithms for the suboptimal feedback control of parabolic systems is presented. Three specific techniques within this class are developed and tested numerically using a linear example with three different types of boundary conditions. Performance of two of these techniques is found to compare favorably with that obtained by use of a rigorous open-loop optimal control computed by the gradient method. It is also shown for linear systems having quadratic performance criteria that the new algorithms are superior to the technique presented by Vermeychuk and Lapidus (1973) in cases where state deviation has a weighting coefficient much greater than that for control energy.

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SCOPE

For purposes of control implementation, many engineering processes are best modeled by a system of partial differential equations. The packed bed tubular reactor is one such example.

The difficulties associated with the implementation of classical optimal control techniques on systems governed by partial differential equations are often so great as to preclude the use of optimal control for such systems, even when the results would be economically advantageous. The specific nature of these difficulties, discussed by Vermeychuk and Lapidus (1973), can be traced to two central causes.

1. The classical optimal control is entirely precomputed using an idealized system model, then applied in feedforward mode. Insensitivity of the resulting control to change in initial conditions, stochastic parameter variations, and modeling errors is a serious problem.

2. The synthesis of any distributed optimal control, either in open loop form or in feedback form, as is possible in linear-quadratic situation, requires a large, general purpose digital computer. This requirement may conceivably be met in a practical situation, but only in very special cases. More realistically, a small process-control type digital computer would be employed in practice.

The previously mentioned work presented examples of control algorithms which produced a near-optimal feedback control for certain classes of distributed systems at the expense of very little computational effort. One of these algorithms is based upon the successive minimization of the kernel of the performance functional. The particular technique considered was only one of a class of such techniques, in which various functions of the performance functional could be used.

This paper further develops the subject of suboptimal feedback control via use of the performance functional kernel by presenting new algorithms in the general class, and by addressing two questions left unanswered in the initial treatment, relating to the stability of the controlled system and the proper choice of the best value for the free parameter appearing in the algorithms.

CONCLUSIONS AND SIGNIFICANCE

Two versions of a suboptimal feedback control algorithm based on the successive instantaneous minimization of the time derivative of the kernel of a quadratic performance functional, as well as an algorithm based on minimization of the kernel itself, are applied to a previously reported one-dimensional linear parabolic system with different types of boundary conditions. Performance of the suboptimal control algorithms was excellent, especially in the attainment of the desired state. Performance indices obtained with the suboptimal techniques were only 12 to 13% larger (typically) than those obtained by application of a rigorous optimal control synthesis procedure.

The techniques developed in this paper are found to be superior to previously developed suboptimal control techniques for cases in which the state deviation term in the quadratic performance functional has a weighting coefficient 50 to 100 times greater than that of the control energy term. For all techniques discussed, performance only slightly worse than optimal is obtained with minimal computational effort and with none of the disadvantages which hinder the application of feedforward control.

The results presented here are significant in that they demonstrate the near optimal control of a class of systems governed by partial differential equations with extremely simple control software. Because this simple software requires only a small computer to utilize it, continued development along these lines will render the direct digital control of complex engineering systems increasingly more attractive from a practical point of view.

The selection of the single scalar free parameter in the suboptimal control techniques is considered, and it is concluded that trial and error is the best method for linear or mildly nonlinear systems.

The stability of linear systems under one type of feedback control discussed herein is investigated and is found to be guaranteed. It is pointed out that stability for nonlinear controlled systems will depend on the individual case under investigation.

THE CONTROL ALGORITHMS

Here we develop a class of suboptimal feedback control algorithms for use with parabolic systems of the form

$$u_t = f(x, t, u, u_x, u_{xx}, v)$$
$$u(x, t_0) = u_0(x) + \text{spatial b.c.}$$
$$v = v(x, t), \text{ a volume control, unconstrained}$$

The performance functional is of a general type, incorporating a fixed final time.

$$J = \int_{t_0}^{t_f} \phi(u, v, t)\,dt$$

To synthesize the classical optimal (that which minimizes J) control for a system of this type, the following two-point boundary value problem must be solved:

$$u_t = f(x, t, u, u_x, u_{xx}, \lambda) \quad (1)$$

$$u(x, t_0) = u_0(x) \qquad + \text{spatial b.c.}$$

$$-\lambda_t = \phi_u + [\nabla_u f]\lambda \quad (2)$$

$$\lambda(x, t_f) = 0 \qquad + \text{spatial b.c.}$$

Here $\lambda(x, t)$ is the adjoint variable used in the definition of the Hamiltonian of the system

$$H = \phi + \langle\lambda, f\rangle \quad (3)$$

The optimal control is given by

$$\nabla_v H = 0 \quad (4)$$

or

$$\phi_v + \langle\lambda, f_v\rangle = 0 \quad (5)$$

In many cases, Equation (5) may be solved to yield $v^o(x, t)$ explicitly as a function of $\lambda^o(x, t)$. The function $\lambda^o(x, t)$ is one of a pair $\lambda^o(x, t) : u^o(x, t)$ which satisfy Equations (1) and (2). The function $u^o(x, t)$ is the optimal trajectory of the system, projected into state space.

It is obvious that the optimal control is not directly attainable as an instantaneous function of the state, except in the case where $f(x, t, u, u_x, u_{xx}, v)$ is linear, and where ϕ is quadratic in both u and v. Then the TPBVP may be transformed into an initial value problem via a Riccati transformation, allowing the generation of $v^o(x, t)$ in a feedback form. Otherwise, before the optimal control is computed, the TPBVP, Equations (1) and (2) must be solved for the entire state and adjoint trajectories. This requires a substantial computational effort. The Riccati equation resulting from the linear-quadratic case cited above is even more complex.

In this as in previous work (Vermeychuk and Lapidus (1973), we revise the criteria for optimality in such a way as to allow the generation of the control in feedback form without excessive sacrifice of performance.

The classical criterion for optimality is

$$\text{minimize} \quad J = \int_{t_0}^{t_f} \phi(u, v, t)\,dt$$

The revised criterion treated in this paper is

$$\text{minimize} \quad \dot{\phi}(u, v, t)\ ;\forall t$$

Although compliance with the revised criterion will not guarantee satisfaction of the classical criterion, near-optimal performance may be reasonably expected, since the suboptimal control will ensure that ϕ is either decreasing as rapidly as possible or increasing as slowly as possible with increasing time.

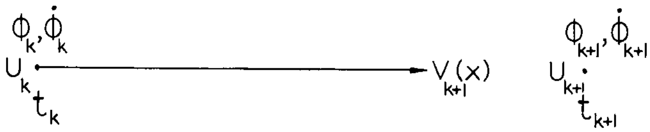

Fig. 1. Schematic representation of suboptimal control algorithms

In the well-known infinite final time linear regulator problem, with quadratic performance functional, $\phi(t)$ will reach an ultimate value of zero if the system is being driven to an equilibrium state. Since, in this case, $\phi(t) \geq 0$ at all t, then min $\dot{\phi}(t) = 0$, which will also be attained. The system will then remain at equilibrium unless a subsequent disturbance occurs.

For regulation to a desired state which is not an equilibrium state, the ultimate value of $\phi(t)$ will be > 0, yet the controller will maintain $\dot{\phi}(t) \leq 0$. Intuitively, then, we would expect our revised criterion to be reasonable.

The essential problem in the development to follow is schematically depicted in Figure 1.

Given a system with state u_k at time t_k, we seek a control $v_{k+1}(x)$ which, when applied to the system over the time interval $t_k < t \leq t_{k+1}$, derives the system to the state u_{k+1} while yielding a minimal value for either $\dot{\phi}_k$ or $\dot{\phi}_{k+1}$, at our option.

Three different specific algorithms to accomplish this are presented, the differences among them resulting from the choice of alternative approximate forms for $\dot{\phi}$.

In what follows the development is carried out for the linear system:

$$u_t = \alpha\, \nabla^2 u + v \quad (6)$$

with a quadratic performance functional kernel of the form

$$\phi(u, v, t) = \mu||u - u_D||^2 + \gamma||v||^2 \quad (7)$$

Generalization of the following algorithms to nonlinear systems and more general forms of ϕ presents no great problem. We have chosen the simple system to maintain clarity of exposition but will indicate in some detail how extensions can be made.

To develop the general form of the suboptimal control algorithms using minimization of $\dot{\phi}_{k+1}$ as a control criterion, we adopt a variational technique. In our case

$$\phi = \mu \langle(u - u_D), \quad (u - u_D)\rangle + \gamma\langle v, v\rangle \quad (8)$$

Forming the time derivative at t_{k+1}

$$\dot{\phi}_{k+1} = 2\mu \left\langle \frac{\partial u_{k+1}}{\partial t},\ (u_{k+1} - u_D) \right\rangle + 2\gamma \left\langle \frac{\partial v_{k+1}}{\partial t},\ v_{k+1} \right\rangle \quad (9)$$

Later, we will introduce various approximations for the time derivatives in the right-hand side of Equation (9), but to preserve generality at this point, let

$$\frac{\partial u_{k+1}}{\partial t} = r_{k+1}\,(u, v) \quad (10a)$$

$$\frac{\partial v_{k+1}}{\partial t} = s_{k+1}\,(u, v) \quad (10b)$$

Note that the approximate forms of the time derivatives may be functions of both the control and state variables. Obtaining the first variation of $\dot{\phi}_{k+1}$ with respect to control and state perturbations yields

$$\delta\phi = 2\mu \left\langle \left[r_u^* \, u + r - r_u^* \, u_D + \frac{\gamma}{\mu} s_u^* \, v \right], \delta u \right\rangle + 2\left\langle \left[s_v^* \, v + s + \frac{\mu}{\gamma} r_v^* \, u \right], \delta v \right\rangle \quad (11)$$

The gradients of r and s with respect to u and v, as well as the adjoints of these gradients, are obtained directly upon specification of forms for r and s. Obviously, δu and δv are related by the state equation. In our case, with a linear parabolic state equation,

$$\delta u_{k+1}(x) = \int_0^\tau \int_\Omega g(x, \tau; \xi, \theta)\delta v_{k+1}(x) \; d\xi d\theta \quad (12)$$

where $g(x, \tau, \xi, \theta)$ is the causal Green's function for the linear parabolic system. In the case of a nonlinear parabolic system, the following relationship applies:

$$\delta u_{k+1}(x) = \int_0^\tau \int_\Omega \hat{g}(x, \tau; \xi, \theta; t_k)\delta v_{k+1}(x) \; d\xi d\theta \quad (13)$$

In this expression, $\hat{g}(x, \tau; \xi, \theta; t_k)$ is the causal Green's function for the parabolic equation obtained by linearization of the state equation about the state and parameter values at $t = t_k$.

Since δv_{k+1} is independent of time, Equation (12) represents an integral transformation of the form

$$\delta u_{k+1}(x) = \int_\Omega K(x, \xi)\delta v_{k+1}(\xi) \; d\xi \quad (14)$$

while Equation (13) represents an integral transformation with a time-dependent kernel

$$\delta u_{k+1}(x) = \int_\Omega \hat{K}(x, \xi, t_k)\delta u_{k+1}(\xi) \; d\xi \quad (15)$$

This is an important distinction, as we shall see later. Returning to the linear case, we can write in shorthand form

$$\delta u = K \, \delta v \quad (16)$$

where K is an integral operator. At this stage, an approximation is introduced. In order to reduce computation time, the integral operator K is approximated by a single scalar multiplier η. Thus, in our approximate form

$$\delta u = \eta \delta v \quad (17)$$

The choice of η is left open, although we know that $\eta > 0$, since it approximates the overall effect of a causal Green's function. Substituting the approximation in Equation (17) into the general condition, Equation (11), yields

$$\mu\eta \left[r_u^* \, u + r - r_u^* \, u_D + \frac{\gamma}{\mu} s_u^* \, v \right] + \left[s_v^* \, v + s + \frac{\mu}{\gamma} r_v^* \, u \right] = 0 \quad (18)$$

the general form of the suboptimal control law. We may now form three specific versions of the suboptimal control law by choosing appropriate forms for r and s.

Version I

In this version, backword difference approximations are used for the time derivates.

$$r \cong \frac{1}{\tau} (u_{k+1} - u_k) \quad (19a)$$

$$s \cong \frac{1}{\tau} (v_{k+1} - v_k) \quad (19b)$$

The time intervals between state measurements are assumed to be equal.

$$\tau = t_{k+1} - t_k \quad (20)$$

The resulting specific control law is

$$v_{k+1}(x) = \frac{1}{2} v_k - \frac{\mu}{2\gamma} \eta \left[(u_{k+1} - u_D) + (u_{k+1} - u_k) \right] \quad (21)$$

Version II

Here, the time derivative of the state is obtained from the state equation itself, evaluated at t_k. The backward difference is retained for the time derivative of the control.

$$r \cong \alpha \nabla^2 u_k + v_{k+1} \quad (22a)$$

$$s \cong \frac{1}{\tau} (v_{k+1} - v_k) \quad (22b)$$

The resulting feedback control law is

$$\left(\mu\eta + \frac{2\gamma}{\tau} \right) v_{k+1}(x) = \frac{\gamma}{\tau} v_k - \mu (u_{k+1} - u_D) - \mu\eta\alpha\nabla^2 u_k \quad (23)$$

Extension to nonlinear state equations is rather simple if the state equation is of the form

$$\frac{\partial u}{\partial t} = f(x, t, u, u_x, u_{xx}) + \psi(x, t, u, u_x, u_{xx})v = f(x) + \psi(x)v \quad (24)$$

The control law becomes

$$\left[\mu\eta \, \psi_k + \frac{2\gamma}{\tau} \right] v_{k+1}(x) = \frac{\gamma}{\tau} v_k - \mu \, \psi_k(u_{k+1} - u_D) - \mu\eta f_k \quad (25)$$

where ψ_k and f_k refer to the functions $f(x)$ and $\psi(x)$ evaluated with respect to time t_k. Note the similarity of the two laws.

Version III

The third algorithm is obtained directly from the first. Recalling the form of the feedback control law in the first algorithm.

$$v_{k+1}(x) = \frac{1}{2} v_k - \frac{\mu}{2\gamma} \eta \left[(u_{k+1} - u_D) + (u_{k+1} - u_k) \right] \quad (26)$$

the $(u_{k+1} - u_k)$ term is replaced by the product of the time interval τ and the time derivative of the state, evaluated at t_k. For the simple case in Equation (6)

$$(u_{k+1} - u_k) \cong \tau \left[\alpha \, \nabla^2 u_k + v_{k+1} \right] \quad (27)$$

resulting in the form of the control law

$$v_{k+1}(x) \left[1 + \frac{\mu\tau}{2\gamma} \eta \right] = \frac{1}{2} v_k - \frac{\mu}{2\gamma} \eta \left[(u_{k+1} - u_D) + \tau\alpha\nabla^2 u_k \right] \quad (28)$$

In the case of a state equation of the form in Equation (24), we have

$$(u_{k+1} - u_k) \simeq \tau \left[f_k + \psi_k \, v_{k+1} \right] \quad (29)$$

This results in a control law of the form

$$\left[1 + \frac{\mu\tau}{2\gamma}\eta\,\psi_k\right] v_{k+1}(x) = \frac{1}{2} v_k - \frac{\mu}{2\gamma}\eta\left[(u_{k+1} - u_D) + \tau f_k\right] \quad (30)$$

All three specific control laws, Equations (21), (23), and (28) have two important features. First, the constant η is arbitrarily chosen. In a practical implementation, η would be adjusted for optimum performance as are the settings on common pneumatic or electrical controllers. Second, the control laws give v_{k+1} as a function of u_{k+1}, which is unknown at time t_k. Therefore, an estimate of $u_{k+1}(x)$ must be made in order to successfully apply the control laws. Since we have some mathematical model of the system to be controlled, we may represent u_{k+1} as a function of u_k and v_{k+1}

$$\hat{u}_{k+1} \cong F(u_k, v_{k+1}) \quad (31)$$

where F represents a numerical integration of the system model from t_k to t_{k+1} with initial condition $u_k(x)$ and constant control $v_{k+1}(x)$. By making use of a simple successive approximation scheme which is an application of the principle of contraction mapping, (Kolmogorov, 1957) the control scheme is executed in the following manner.

1. The state at time t_k, that is, $u_k(x)$, is obtained via an appropriate ensemble of sensors, filters, etc.
2. An initial gross approximation of $v_{k+1}(x)$ is made

$$\hat{v}^{0}_{k+1}(x) = v_k(x)$$

3. Using $\hat{v}^{0}_{k+1}$ as a starting point, the state equation and control equation are applied alternately for N iterations.

$$\hat{u}^{i}_{k+1}(x) = F(u_k, \hat{v}^{i}_{k+1}) \quad (32)$$

$$\hat{v}^{i+1}_{k+1}(x) = \Gamma(u_k, \hat{u}^{i}_{k+1}, u_D, v_k) \quad (33)$$

$$i = 0, 1, \ldots. N-1$$

In each case, Γ is given by the specific control law

4. The control

$$v_{k+1}(x) = \hat{v}^{N}_{k+1}(x) \quad (34)$$

is applied to the systems as a feedback control.

Our computational experience has shown that $N = 2$ is sufficient. All results described in this paper were obtained using $N = 2$.

This scheme of execution is the same for all three versions of the control law. The function Γ will be different for each version, however. All three versions are applicable to either linear or nonlinear systems with any type of boundary conditions. The only requirement is for a model of the system amenable to direct numerical integration.

COMMENTS ON THE ALGORITHMS

Before proceeding to consider the results of computational trials of the previously described algorithms, we will discuss and compare their features. In all three versions, the time derivative of the control variable at $t = t_{k+1}$ was represented as a backward difference form, Equation (19). Since the control, although a function of the spatial coordinates, is a piecewise constant function of time, a backward difference is the only reasonable approximation to a true derivative which is either zero or does not exist. In the case of the time derivative of the state, three different approximations are employed. For Version I, the backward difference is used. In Version II, the state time derivation at t_{k+1} is represented by the value of this derivative at t_k. In version III, the difference between the state at two successive sampling points is represented by the well-known Euler approximation form.

Subjectively speaking, the approximation involved in version II is more gross than the corresponding approximations in either version I or III. This judgement is supported by the clearly inferior performance of version II we obtained in our computational investigation.

In the preceding derivations and in the examples to follow, primary emphasis is placed on the case of the linear system. Fully nonlinear problems may be treated quite simply, of course, via approximate modification of the system model used to generate estimates of $u_{k+1}(x)$ in the control algorithms. A linearized model would be substituted, whose parameters would be computed as functions of x, t_k, and $u_k(x)$. In cases such as these, variations in performance would be expected, depending on the faithfulness of the model in the control algorithm. However, the feedback nature of these algorithms would provide some compensation for an incorrectly chosen system model.

Finally, nothing has been said about how the state data are to be obtained. In many cases, state measurements will be corrupted by noise, and appropriate filtering procedures will be required to render the state measurements suitable for input into the control algorithms previously described. The combined performance of the state estimator and controller is the goal of primary importance; however, in an early developmental stage we wish to reject any control algorithms which give poor performance in the deterministic situation. Therefore, this work is concerned only with the deterministic case.

RESULTS

The results obtained from application of the new control algorithms are presented in this section, together with results obtained via the use of a classical optimal volume control, synthesized by the gradient method of Seinfeld and Lapidus (1968). The classical optimal control serves as a standard of comparison. The scheme based on the minimization of ϕ presented by Vermeychuk and Lapidus (1973) is also applied to the example for purposes of comparison.

The example problem is a simple linear parabolic system in one dimension with volume control. The performance functional is a weighted quadratic, and the three principal types of boundary conditions are considered in turn.

System:

$$\frac{\partial}{\partial t} u(x,t) = \alpha \frac{\partial^2}{\partial x^2} u(x,t) + v(x,t) \quad (35)$$

$$x \in (0,1) \qquad t \in [0, t_f]$$

Initial Condition:

$$u(x,0) = u_0(x) \quad (36)$$

Boundary Conditions:

1st kind $\quad u(0,t) = u(1,t) = 0 \quad (37)$

2nd kind $\quad \frac{\partial}{\partial x} u(0,t) = \frac{\partial}{\partial x} u(1,t) = 0 \quad (38)$

3rd kind $$\frac{\partial}{\partial x} u(0,t) - \beta u(0,t) = 0 \quad (39)$$

$$\frac{\partial}{\partial x} u(1,t) + \beta u(1,t) = 0 \quad (40)$$

Performance functional

$$J = \int_0^{t_f} \int_0^1 \mu\,(u - u_D)^2 + \gamma\, v^2\, dx dt \quad (41)$$

Problem conditions

$u_0(x) = 0.0$ $\qquad$ $u_D(x) = 1.00$
$\alpha = 0.10$ $\qquad$ $\beta = 1.00$
$\mu = 1.00$ $\qquad$ $t_f = 1.00$ or 2.00

Application of the various suboptimal control schemes in the previously explained manner, and application of the gradient method in the manner presented by Vermeychuk and Lapidus (1973) yielded the following table of performance indices. In all cases where a value of η for a suboptimal control algorithm is quoted, this value was found to yield the best performance.

TABLE 1. COMPARATIVE PERFORMANCE OF CONTROL ALGORITHMS

Algorithm	t_f	$\gamma = 0.10$ B.C. Type/η value 1	2	3
Gradient	1.0	0.53231	0.33230	0.35580
	2.0	0.79194	0.33399	0.36846
min ϕ	1.0	0.60565/0.20	0.35528/0.30	0.37745/0.30
	2.0	0.93724/0.25	0.36600/0.30	0.39004/0.30
min $\dot{\phi}$	1.0	0.61657/0.25	0.36146/0.40	0.38229/0.40
Ver. I	2.0	0.94721/0.25	0.37755/0.40	0.39355/0.40
min $\dot{\phi}$	1.0	0.61863/0.25	0.37696/0.35	0.39676/0.35
Ver. III	2.0	0.94963/0.25	0.42162/0.30	0.42770/0.30

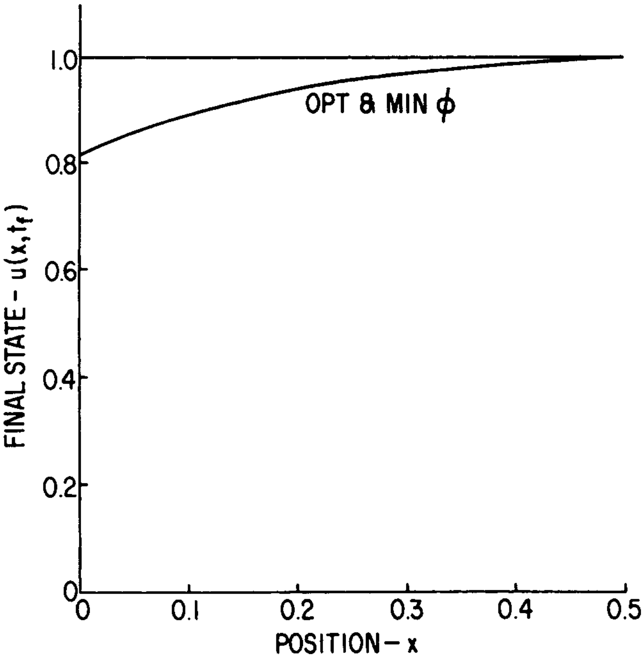

Fig. 3. Final state vs. position for B.C. of 3rd kind: Optimal and min ϕ suboptimal control

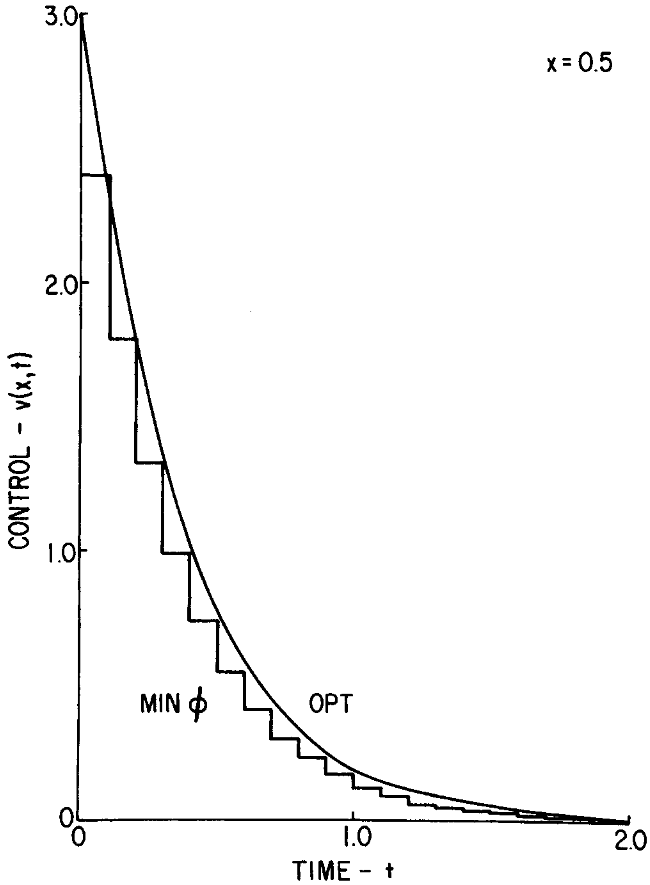

Fig. 2. Control vs. time for B.C. of 3rd kind; Optimal and min ϕ suboptimal with $\eta = 0.30$

The data in Table 1 was obtained using a value of $\gamma = 0.10$ in Equation (41). It can be seen that the min ϕ form of suboptimal feedback control performs very well. The performance indices obtained under min ϕ control exceed the absolute optimal performance indices by an average of only 11.6%. In view of the great reduction in complexity and added advantages of feedback operation, the min ϕ scheme appears to be quite acceptable as an alternative to classical optimal control in this case.

In Figures 2 and 3, previously cited results for suboptimal feedback control of the min ϕ type are also plotted. Referring to Figure 3, the final state of the system is essentially the same under both types of control. This is of great importance if the performance criterion contains a final-state deviation term. The control versus time curves, appearing in Figure 2, are remarkably similar. The suboptimal feedback control is a piecewise constant function of time and is represented as such. Close agreement between optimal and suboptimal control is observed, however.

Further examination of the table shows that the min $\dot{\phi}$—Version I control is an acceptable alternative to optimal control in this case. The performance indices obtained exceed the optimal performance indices by an average 13.2%.

Figures 4 and 5 describe the action of this suboptimal control scheme graphically. Note the closeness of the final state to the desired state. These figures are taken from the system with $t_f = 2.0$ and boundary conditions of the 3rd kind. All cases, however, exhibited similar performance.

The third version of the min $\dot{\phi}$ algorithm performed a bit worse than the first version, yielding performance indices some 3.8% larger. Under identical problem conditions, the final state versus position and control versus time profiles were very similar to those of the first version, shown in Figures 4 and 5.

Although the data in Table 1 show acceptable performance for the min $\dot{\phi}$ algorithms, performance of the min ϕ algorithm is better. Table 2, however, exhibits typical data which shows the superiority of the min $\dot{\phi}$ over the min ϕ algorithm under certain conditions. As a representative case, the linear example with boundary conditions of the third kind and $t_f = 1.0$ was chosen. It can be seen that for small values of γ, the min $\dot{\phi}$ algorithm delivered better performance than the min ϕ algorithm. The values of η recorded with the performance indices produced the best performance in each case. Values of γ as small as 0.01 or 0.02 would seem reasonable to expect in practical situations such as the control of heating of massive bodies, since the economic penalties associated with state deviations may often be much greater than the cost of control, though the cost of control is never zero. Therefore, the min $\dot{\phi}$ algorithm emerges as the technique of choice in such cases. In situations where cost of control and state deviations are of the same order of magnitude, the min ϕ algorithm is preferable.

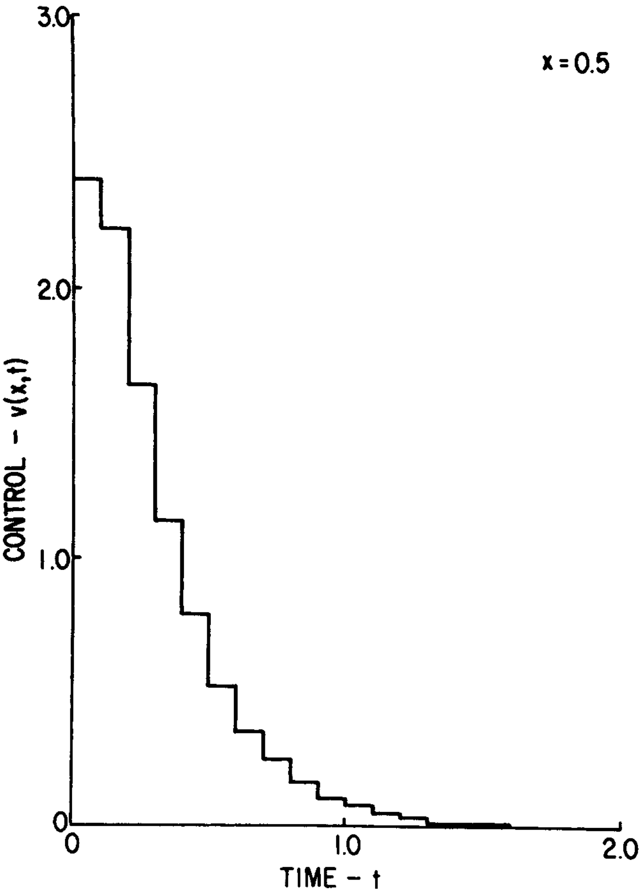

Fig. 4. Control vs. time for B.C. of 3rd kind; min $\dot{\phi}$ Suboptimal control, version I, $\eta = 0.40$

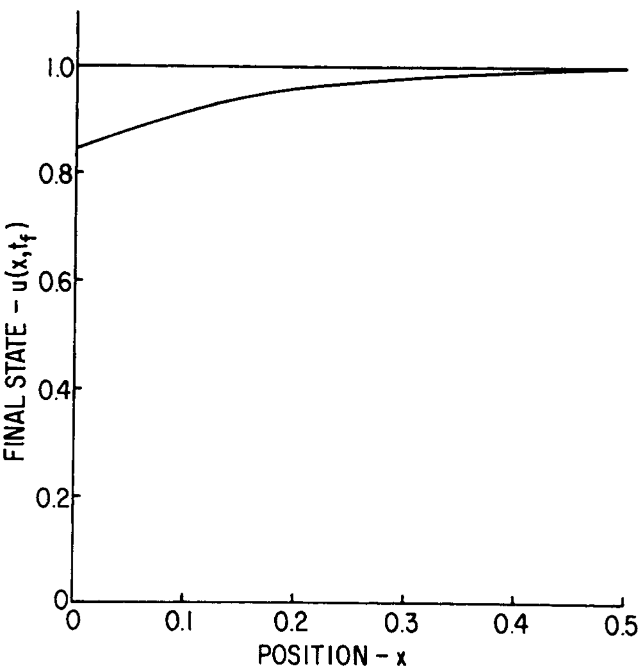

Fig. 5. Final state vs. position for B.C. or 3rd kind; min $\dot{\phi}$ Suboptimal control, version I, $\eta = 0.40$

TABLE 2. EFFECT OF CONTROL ERROR WEIGHTING ON SUBOPTIMAL CONTROL PERFORMANCE

Linear parabolic system, b.c. of 3rd kind, $T_f = 1.0$

Algorithm

γ	Gradient	Min ϕ	Min $\dot{\phi}$, Ver. I
1.00	0.80400	0.82966/0.50	0.84191/0.50
0.10	0.35580	0.37745/0.30	0.38229/0.40
0.02	0.15799	0.20757/0.20	0.19103/0.20
0.01	0.11174	0.16643/0.15	0.14513/0.15

TABLE 3. EFFECT OF η ON PERFORMANCE OF MIN ϕ CONTROL

η	J	η	J
0.05	0.78951	0.30	0.67578
0.15	0.62511	0.35	0.65403
0.20	0.60565*	0.50	0.78011
0.25	0.60823	0.75	1.06783

COMMENT

From the data presented here, it is clear that either the min ϕ or min $\dot{\phi}$ algorithms are acceptable alternatives to optimal control for linear parabolic systems, offering the advantages of simplicity and feedback operation. The choice between the suboptimal control schemes is dictated by the relative weighting of state deviations and control cost in the measure of system performance. For systems having boundary conditions of the second or third kind, attainment of the desired state is especially good under suboptimal control.

FURTHER CONSIDERATION OF THE MIN ϕ METHOD

In the original presentation of the min ϕ technique, two questions were not answered. They are stated and answered here.

1. Can the best value of η be chosen a priori?

For linear systems, and perhaps for some nonlinear systems, trial and error choice of η seems a logical course of action for 2 reasons:

A. The value of η cannot be obtained with complete certainty by analytical methods.

B. The choice of η, or $K(x, \xi)$, the integral operator kernel which η approximates, in full accordance with their definition, will not guarantee the most satisfactory system performance, since the min ϕ method does not satisfy the principle of optimality.

These points will be elaborated in what follows.

2. Given that the free or unforced system is stable, will the controlled system be stable as well?

For a linear parabolic system under min ϕ control, $\phi(t)$ may be represented as a Lyapunov function in the state of the controlled system. For cases with boundary conditions of the second or third kind, the controlled system is regionally asymptotically stable in the large about the desired state. For cases with boundary conditions of the first kind, the theoretical existence of a finite offset in the suboptimal controller does not seem to adversely affect attainment of the desired state in our numerical trials.

Returning briefly to the first question, it can be shown that the rigorously correct relationship between δu and δv is

$$\delta u_{k+1}(x) = \int_\Omega K(x, \xi)\ \delta v_{k+1}(\xi) d\xi \qquad (14)$$

where

$$K(x, \xi) = \int_0^\tau g(x, \tau; \xi\theta) d\theta \qquad (42)$$

If the integral operator in Equation (14) is approximated by a scalar multiplier,

$$\delta u = \eta \delta v \qquad (17)$$

the only statement that can be made about η is

$$\eta \leq \max_{\varphi(x)} \frac{<\varphi, k\varphi>}{<\varphi, \varphi>} \qquad (43)$$

This Rayleigh quotient takes on the value of the maximum eigenvalue of K for the proper choice of $\varphi(x)$. Thus, we know very little about the definite value of η. In a nonlinear case, η would depend on time, as well as the system, boundary conditions, and length of the sampling interval. Note that if the min ϕ control law were written in a more rigorous form involving K,

$$v_{k+1}(x) = -\frac{\mu}{\gamma} \int_\Omega K(x, \xi)\ [u_{k+1}(\xi) - u_D(\xi)]\ d\xi \qquad (44)$$

Then $K(x, \xi)$ would have to be determined at the expense of considerable computational effort, especially for nonlinear cases, where K is time dependent.

For linear cases, then, it would seem reasonable to search for the best value of η by trial and error. Table 3 illustrates the considerable effect of the value of η on the performance of the min ϕ method. The example was linear with boundary conditions of the first kind. A value of $\gamma = 0.10$ was used, which gave an optimal performance index of $J_0 = 0.53231$, as obtained with the gradient method.

With regard to stability under min ϕ control, it can be shown that for a linear parabolic system [Equation (35)] under min ϕ control, ϕ and $\dot{\phi}$ are given by

$$\phi(t) = \mu\left(1 + \mu\frac{\eta^2}{\gamma^2}\right) <u, u> \qquad (45)$$

$$\dot{\phi}(t) = 2\mu\left(1 + \mu\frac{\eta^2}{\gamma^2}\right)\left[<\nabla^2 u, u> - \frac{\eta}{\gamma}\eta <u, u> \right] \qquad (46)$$

Note that $\phi(t) > 0$ and $\dot{\phi}(t) < 0$ for $u(x, t) \neq 0$ and $\eta > 0$. Thus, $\phi(t)$ is a Lyapunov function. Assuming that the system is normalized to make the desired state equal to zero, then $\dot{\phi}(t) = 0$ when

$$\left[<\nabla^2 u, u> - \frac{\mu}{\gamma}\eta <u, u> \right] = 0 \qquad (47)$$

For boundary conditions of the second or third kind, the stable state corresponds to the desired state. For boundary conditions of the first kind, the stable state is given by the solution of

$$\frac{\partial^2}{\partial x^2} u(x) - \frac{\mu}{\gamma}\eta\ (u(x) - u_D) = 0 \qquad (48)$$

The solution of (48) is not necessarily u_D, but we have encountered no problems with our numerical experiments with the min ϕ algorithm applied to systems with Dirichlet boundary conditions.

For nonlinear systems, no general statement regarding stability may be made. The functions $\phi(t)$ must be formed and investigated for each separate case. If $\phi(t)$ satisfies the requirements of a Lyapunov function, stability of the controlled system about some state is guaranteed.

NOTATION

F = function
H = Hamiltonian
J = performance index
K = integral operator or kernel
N = maximum no. of iterations in suboptimal control algorithms
Q = weighting kernel in a quadratic performance functional
Q^* = adjoint of Q
f = general function representing the right-hand side of a parabolic PDE
g = causal Green's function of a linear parabolic system
$\hat{g}$ = causal Green's function of a linearized parabolic system
r = approximate form of time derivative of the state variable
s = approximate form of time derivative of the control variable
t = time
t_0 = initial time
t_f = final time
u = state
u_0 = initial state
u_D = desired state
$\hat{u}$ = approximate value of state
v = volume control
v^0 = optimal control
$\hat{v}$ = approximate value of control
x = spatial variable

Greek Letters

Γ = function representing a suboptimal control algorithm
Ω = spatial domain
α = constant
β = constant
γ = constant
δ = variation operator
λ = adjoint variable
μ = constant
τ = time interval

ϕ = kernel of performance functional
$\varphi(x)$ = arbitrary function of x
$\psi(x)$ = arbitrary function of x

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Theoretical Development and Experimental Verification of a Novel, Well-Mixed Vessel

The mixing characteristics of a vessel containing no moving parts have been studied theoretically and experimentally. The vessel consists of two chambers separated by a porous barrier. Mixing results because elements of fluid permeating the barrier at various distances from the inlet reside for different periods of time within the vessel and combine with other elements having entered earlier and later. An apparatus was designed a priori and experimentally verified to give a residence-time distribution function the same as a completely mixed vessel. The method was extended to show that in principle a vessel exhibiting any residence-time distribution function can be designed by modifying the geometries of the chambers and the porous barrier.

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SCOPE

Well-mixed vessels are widely used in production, pilot plant, and research applications and in their most common form these systems consist of a container and an efficient stirrer. The theory and performance of such systems has been extensively studied and much attention has been given to testing how closely a given system approaches complete mixedness. One commonly applied test is to observe the transient tracer concentration in the effluent stream of a steady flow system when the tracer is introduced as a step-function in the input stream. This test permits the determination of the residence-time distribution function (RTDF) which can be used to evaluate how closely the performance of a given vessel approaches that of a completely-mixed vessel.

This paper concerns a new kind of mixing vessel which is novel in that it contains no moving parts. The new mixing concept appears to offer three distinct advantages over its older counterpart:

1. It eliminates the need for a stirrer-drive system and its attendant rotary seal.
2. It permits a priori design of any mixing behavior. More specifically, it can in principle be designed to give any predetermined RTDF.
3. It permits mixing without the dissipation of high mechanical energy inputs.

The basic principle upon which the new mixing vessel operates is to regulate the amount of flow through sections of a high-resistance porous barrier at progressively greater distances from the input and output of the vessel. In this way, the residence time of each fraction of the input stream can be controlled to predetermine the RTDF. The completely mixed vessel will be the main concern of this paper from the theoretical and experimental point of view; however, some effort will be made to generalize the theoretical treatment. Although the experimental work has been limited to gas-gas systems, the theory has no explicit limitations to gases and would apply to liquids. However, extremely high viscosities would probably invalidate the model.

It should be made clear at the outset that the completely-mixed vessel as obtained by a stirrer in a container is not mechanistically equivalent in its mixing properties to the completely mixed vessel obtained using the barrier. This derives from the well-known fact that two systems can display identical RTDF's but vary greatly in the detailed mechanisms which give rise to them. Therefore, the two systems are equivalent only with respect to first-order processes, that is, systems in which events depend uniquely upon time but not position in the vessel. Obviously, many systems do not meet this restriction. For example, the two vessels, although each had the same RTDF as a completely mixed vessel, would differ greatly in the conversion of a second-order chemical reaction for the same nominal residence time. This, perhaps, offers some interesting possibilities for the barrier mixer in a homogeneous or a heterogeneous reactor configuration. However, only first-order processes will be considered in this paper.

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